Journal of Statistical Physics, Vol. 54, Nos. 1/2, 1989

Absence of Symmetry Breaking for Systems of Rotors with Random Interactions

Cezar A. Bonato[1] **and Massimo Campanino**[2]

Received April 29, 1988; revision received August 15, 1988

We prove that Gibbs states for the Hamiltonian $H = -\sum_{xy} \tilde{J}_{xy} s_x \cdot s_y$, with the s_x varying on the N-dimensional unit sphere, obtained with nonrandom boundary conditions (in a suitable sense), are almost surely rotationally invariant if $\tilde{J}_{xy} = J_{xy}/|x-y|^\alpha$ with J_{xy} i.i.d. bounded random variables with zero average, $\alpha \geqslant 1$ in one dimension, and $\alpha \geqslant 2$ in two dimensions.

KEY WORDS: Disordered systems; Gibbs states; symmetry breaking.

1. INTRODUCTION

A typical Hamiltonian for a system of rotors with long-range random interaction is

$$H = -\sum_{xy} J_{xy} |x-y|^{-\alpha} s_x \cdot s_y \tag{1.1}$$

where J_{xy} are i.i.d. random variables with zero average and s_x takes values on the N-dimensional unit sphere. For these systems Picco[5] found the absence of symmetry breaking for $\alpha > 3/2$ in one dimension and $\alpha > 3$ in two dimensions.

Van Enter and Fröhlich[3,4] developed methods to study the case $\alpha > 1$ in one dimension and $\alpha > 2$ in two dimensions. Here we obtain the absence of symmetry breaking for a class of models that include (1.1) with $\alpha \geqslant 1$ in one dimension and $\alpha \geqslant 2$ in two dimensions, provided one considers Gibbs states obtained with nonrandom boundary conditions (in a sense to be precisely defined). For the same model with discrete symmetry (random

[1] Departamento de Fisica, Universidade Federal da Paraiba, João Pessosa, Pb, Brazil.
[2] Istituto di Matematica, Universita' della Basilicata, Potenza, Italy.

Ising model) the absence of spin flip symmetry breaking (and in general the absence of phase transitions) in the same hypotheses has been proved for $\alpha > 1$ in one dimension but is not believed to hold for $\alpha = 1$. The reasons why we must impose the restriction on the boundary conditions in the region $1 \leqslant \alpha \leqslant 3/2$ for $d = 1$ and $2 \leqslant \alpha \leqslant 3$ for $d = 2$ are the same as in the case of the one-dimensional random Ising model[2] and are explained in the introduction of ref. 2. We cannot exclude rotational symmetry breaking for states obtained from boundary conditions dependent on the realization of the interaction.

2. RESULTS

For N a fixed, positive integer, let S_N be the N-dimensional unit sphere $S_N = \{x \in R^{N+1} \mid \|x\| = 1\}$. For $\Lambda \subset \mathbb{Z}^d$ (we shall be only concerned with the cases $d = 1$ and $d = 2$) our configuration space in the volume Λ will be $\mathscr{S}_\Lambda = S_N^\Lambda$. If $\Lambda_1 \subset \Lambda_2$ and $s \in \mathscr{S}_{\Lambda_2}$, then $s|_{\Lambda_1}$ will denote the restriction of s to Λ_1. For each unordered pair x, $y \in \mathbb{Z}^d$, $x \neq y$, let $\tilde{J}_{xy} \in \mathbb{R}$,

$$\tilde{J}_{xy} = |x - y|^{-\alpha} J_{xy}$$

with J_{xy} uniformly bounded, $\alpha \geqslant 1$ for $d = 1$ and $\alpha \geqslant 2$ for $d = 2$. We define the energy $H_\Lambda(s)$ for Λ finite $\subset \mathbb{Z}^d$ and $s \in \mathscr{S}_\Lambda$ by

$$H_\Lambda(s) = - \sum_{x,\, y \in \Lambda,\, x \neq y} \tilde{J}_{xy} s_x \cdot s_y \tag{2.1}$$

where $s_x \cdot s_y$ denotes the scalar product of s_x and s_y. Given Λ_1 and Λ_2 finite subsets of $\mathbb{Z}^d$, with $\Lambda_1 \cap \Lambda_2 = \varnothing$, we define the interaction $W_{\Lambda_1, \Lambda_2}(s^{(1)}, s^{(2)})$ by

$$W_{\Lambda_1, \Lambda_2}(s^{(1)}, s^{(2)}) = - \sum_{x \in \Lambda_1,\, y \in \Lambda_2} \tilde{J}_{xy} s_x^{(1)} \cdot s_y^{(2)} \tag{2.2}$$

When the $\tilde{J}_{xy}$ decay sufficiently fast with $|x - y|$ the interaction ($\alpha > 1$ for $d = 1$ and $\alpha > 2$ for $d = 2$) is defined also when one of the two volumes Λ_1, Λ_2 is in finite.

Let $s \in \mathscr{S}_{\mathbb{Z}^d}$ and let C_n be the cube with center at zero and side $2n + 1$,

$$C_n = \{x \in \mathbb{Z}^d \mid |x_i| \leqslant n,\ 1 \leqslant i \leqslant d\}$$

Given $0 < n_1 < n_2$ and given a configuration $\bar{s} \in \mathscr{S}_{C_{n_2}}$, we can define the Gibbs measure in the volume C_{n_1} with boundary conditions $\bar{s}$ in $C_{n_2} \backslash C_{n_1}$ by

$$\mu_{n_1 n_2 \bar{s}}(\phi) = Z_{n_1 n_2 \bar{s}}^{-1} \int \phi(s) \exp[-H_{C_{n_1}}(s) - W_{C_{n_1}, C_{n_2} \backslash C_{n_1}}(s, \bar{s}|_{C_{n_2} \backslash C_{n_1}})]\, d_{C_{n_1}} s$$

for ϕ continuous of $\mathscr{S}_{C_{n_1}}$, where

$$d_{C_{n_1}} s = \prod_{i \in C_{n_1}} ds_i$$

and $Z_{n_1 n_2 \bar{s}}$ is the normalizing constant,

$$Z_{n_1 n_2 \bar{s}} = \int \exp[-H_{C_{n_1}}(s) - W_{C_{n_1}, C_{n_2} \backslash C_{n_2}}(s, \bar{s}|_{C_{n_2} \backslash C_{n_1}})]\, d_{C_{n_1}} s$$

When $\tilde{J}_{xy}$ decays sufficiently fast with the distance between x and y we may take $n_2 = \infty$ in the above formulas, but this is not possible for $\alpha = 1$ in one dimension or $\alpha = 2$ in two dimensions. In the future we shall treat the case $N = 1$. The extension to arbitrary N is immediate. In the case $N = 1$ we can put $s_i = e^{i\theta_i}$ and we shall use as variable θ_i with $0 \leqslant \theta_i < 2\pi$. The sum of two angles will be understood modulo 2π. Let $C(\mathscr{S}_\Lambda)$ be the space of real-valued functions on $\mathscr{S}_\Lambda$, continuous with respect to the product topology. An element A of $C(\mathscr{S}_{\mathbb{Z}^d})$ is called a *local observable* if it depends only on a finite number of coordinates and can therefore be identified with an element of $C(\mathscr{S}_\Lambda)$ for some finite $\Lambda \subset \mathbb{Z}^d$; in this case we shall say that the support of A is contained in Λ. A *state* μ in the infinite volume, i.e., a probability measure on $\mathscr{S}_{\mathbb{Z}^d}$, is said to be *rotationally invariant* if

$$\mu(\sigma_t A) = \mu(A) \tag{2.3}$$

for every $A \in C(\mathscr{S}_{\mathbb{Z}^d})$, where for $t \in \mathbb{R}$, $\sigma_t A$ is the observable obtained from A by rotation of all the angles by t. For every local observable B we have

$$\frac{d}{dt} \mu(\sigma_t B)\bigg|_{t=u} = \frac{d}{dt} \mu(\sigma_t A)|_{t=0} \qquad \text{for} \quad A = \sigma_u B$$

Therefore in order to verify that a state μ is rotationally invariant, it is enough to check that for every local observable A

$$\frac{d}{dt} \mu(\sigma_t A)\bigg|_{t=0} = 0 \tag{2.4}$$

Given an observable $A \in C(\mathscr{S}_\Lambda)$ and a real-valued function f defined on Λ, we define the observable $\sigma * f(f) A$ by

$$\sigma * (f)\, A(\theta) = A(\sigma(f)\theta) \tag{2.5}$$

where $(\sigma(f)\theta)_x = \theta_x + f(x) \pmod{2\pi}$.

Let $\mu_{n_1, n_1, \theta}$ be the finite-volume Gibbs state in the volume C_{n_1} with boundary condition θ in $C_{n_2} \backslash C_{n_1}$. By applying the Schwarz inequality and simple changes of variables in the definition of Gibbs states, we get the following inequality (a particular case of Bogoliubov's inequality; see, e.g., ref. 1):

$$\mu_{n_1,n_2,\theta}\left[\frac{d}{ds}(\sigma * (tf)A\bigg|_{t=0}\right]^2$$

$$\leqslant \mu_{n_1,n_2,\theta}(A^2)\mu_{n_1,n_2,\theta}\left[\frac{d^2}{dt^2}\sigma * (tf)(H_{C_{n_1}}W_{C_{n_1},C_{n_2}\setminus C_{n_1}})\bigg|_{t=0}\right] \tag{2.6}$$

In the following we shall make use of the estimates contained in the following lemma:

Lemma 2.1. There exists a constant C such that for arbitrary n_1, n_2 $(n_1 < n_2)$, $\bar{\theta} \in \mathscr{S}_{\mathbb{Z}}$, and for $x \in C_{n_1}$, $y \in C_{n_1}$ we have

$$\mathbb{E}(\mu_{n_1,n_2,\bar{\theta}}[\cos(\theta_x - \theta_y)\tilde{J}_{xy})]) \leqslant C\|\tilde{J}_{xy}\|_\infty^2 \tag{2.7}$$

where $\|\tilde{J}_{xy}\|_\infty$ is the supremum of $|\tilde{J}_{xy}|$ as a random variable.

Moreover, for $x \in C_{n_1}$, $y \in C_{n_2}\setminus C_{n_1}$

$$\mathbb{E}(\mu_{n_1,n_2,\bar{\theta}}[\cos(\theta_x - \bar{\theta}_y)\tilde{J}_{xy}]) \leqslant C\|\tilde{J}_{xy}\|_\infty^2 \tag{2.8}$$

Proof. We can use the arguments for the analogous bounds in refs. 3 and 4. We only remark that this is correct since we are dealing with finite-volume Gibbs states with fixed (i.e., nonrandom) boundary conditions. We write

$$\mu_{n_1,n_2,\bar{\theta}}(\cos(\theta_x - \theta_y)) = \frac{\tilde{\mu}_{n_1,n_2,\bar{\theta}}(\cos(\theta_x - \theta_y)\exp[\tilde{J}_{xy}\cos(\theta_x - \theta_y)])}{\tilde{\mu}_{n_1,n_2,\bar{\theta}}(\exp[\tilde{J}_{xy}\cos(\theta_x - \theta_y)])} \tag{2.9}$$

where $\tilde{\mu}_{n_1,n_2,\bar{\theta}}$ is the Gibbs state with the same Hamiltonian as $\mu_{n_1,n_2,\bar{\theta}}$ except for the interaction between the sites x and y that is put equal to zero. If the distance between x and y is sufficiently large, we can develop the rhs of (2.9) in power series of $\tilde{J}_{xy}$ and write

$$\mu_{n_1,n_2,\bar{\theta}}(\cos(\theta_x - \theta_y)) = \tilde{\mu}_{n_1,n_2,\bar{\theta}}(\cos(\theta_x - \theta_y)) + \tilde{J}_{xy}C_1(\underline{J}) \tag{2.10}$$

where $C_1(\underline{J})$ is a function of the interactions of all the interactions J_{xy} for x, y in C_{n_2} bounded uniformly in x and y by a constant C. By using (2.10) we get immediately (2.7), since $\tilde{J}_{xy}$ has zero average and the first term on the rhs of (2.10) is independent of $\tilde{J}_{xy}$. By possibly changing the value of the constant, we obtain the inequality for every x and y. Relation (2.8) is obtained in the same way. Here, as in the following, we take the convention to use the same letter C to indicate possibly different constants. ∎

We are now in the position to prove the following.

Theorem 2.2. Assume that $\mathbb{E}(\tilde{J}_{xy})=0$ and that $\|\tilde{J}_{xy}\|_\infty \leqslant \text{const}\cdot |x-y|^{-\alpha}$ with $\alpha \geqslant 1$ for $d=1$ and $\alpha \geqslant 2$ for $d=2$. Then we can find a suitable sequence $n_i \uparrow \infty$ such that for every boundary condition $\bar{\theta} \in \mathscr{S}_{\mathbb{Z}^d}$ and every sequence $\bar{n}_i > n_i$ we have that, with probability one with respect to the realization of the interaction $\{\tilde{J}_{xy}\}$, every state obtained from a convergent subsequence of the sequence $\mu_{n_i,\bar{n}_i,\bar{\theta}}$ is rotationally invariant.

Proof. Let A be an observable with support in a finite region Λ. Given two positive integers n and $\bar{n}$, $n<\bar{n}$, and a boundary condition $\bar{\theta}$, we want to estimate

$$\frac{d}{dt}\mu_{n,\bar{n},\bar{\theta}}(\sigma_t A)\bigg|_{t=0} \tag{2.11}$$

Let f be a real-valued function defined in C_n such that $f(x)=1$ for $x\in\Lambda$ (we are assuming that $\Lambda\subset C_n$). Then we have $\sigma_t A=\sigma * (tf)A$ [see (2.5)]. By applying Bogoliubov's inequality, we get that

$$\left[\frac{d}{dt}\mu_{n,\bar{n},\bar{\theta}}(\sigma_t A)\bigg|_{t=0}\right]^2 \leqslant \mu_{n,\bar{n},\bar{\theta}}(A^2)\,\mu_{n,\bar{n},\bar{\theta}}\left(\frac{d^2}{dt^2}[\sigma * (tf)(H_{C_n}+W_{C_n,C_{\bar{n}}\setminus C_n})]\bigg|_{t=0}\right) \tag{2.12}$$

On the other hand, we have that

$$\begin{aligned}\mu_{n,\bar{n},\bar{\theta}}&\left(\frac{d^2}{dt^2}[\sigma * (tf\rangle\langle H_{C_n}+W_{C_n,C_{\bar{n}}\setminus C_n})]\bigg|_{t=0}\right)\\ &=\sum_{x,\,y\in C_n}\tilde{J}_{xy}[f(x)-f(y)]^2\,\mu_{n,\bar{n},\bar{\theta}}(\cos(\theta_x-\theta_y)\\ &\quad+\sum_{x\in C_n}f(x)^2\sum_{y\in C_{\bar{n}}\setminus C_n}\tilde{J}_{xy}\mu_{n,\bar{n},\bar{\theta}}(\cos(\theta_x-\bar{\theta}_y))\end{aligned} \tag{2.13}$$

We shall aply Lemma 2.1 to estimate the expectations of the correlation functions. We get

$$\begin{aligned}\mathbb{E}(\mu_{n,\bar{n},\bar{\theta}}&\left(\frac{d^2}{dt^2}[\sigma * (tf)(H_{C_n}+W_{C_n,C_{\bar{n}}\setminus C_n})]\bigg|_{t=0}\right)\\ &\leqslant\sum_{x,\,y\in C_n}[f(x)-f(y)]^2\frac{C}{|x-y|^{2\alpha}}+\sum_{x\in C_n}f(x)^2\sum_{y\in C_{\bar{n}}\setminus C_n}\frac{C}{|x-y|^{2\alpha}}\end{aligned} \tag{2.14}$$

We shall see that for a sequence n_i tending sufficiently fast to infinity we can find a sequence of functions $f^{(i)}$ defined on $\mathbb{Z}^d$ such that:

(i) There is a sequence $\tilde{n}_i$, $\tilde{n}_i \uparrow \infty$ and $\tilde{n}_i < n_i$, such that $f^{(i)}(x) = 1$ for $x \in C_{\tilde{n}_i}$.

(ii) The following condition holds:

$$\sum_{i=1}^{\infty} \mathbb{E}\left(\mu_{n_i, \tilde{n}_i, \theta}\left(\frac{d^2}{dt^2}\left[\delta * (tf)(H_{C_{n_i}} + W_{C_{n_i}, C_{\tilde{n}_i} \setminus C_{n_i}})\right]\Big|_{y=0}\right)\right) < \infty \tag{2.15}$$

This implies by (2.12) that for almost every realization of the interaction, if μ is an infinite-volume state obtained as a limit of a convergent subsequence of $\mu_{n_i, \tilde{n}_i, \theta}$ and A is a local observable, then

$$\frac{d}{dt}\mu(\sigma(t)A)\Big|_{t=0} = 0 \tag{2.16}$$

i.e., μ is rotationally invariant.

Let us now make the choice of the functions $f^{(i)}$. As in ref. 1, we can define

$$E(k) = \sum_{x \in \mathbb{Z}^d \setminus \{0\}} (1 - \cos k \cdot x)\, |x|^{-2\alpha}$$

for $k \in [-\pi, \pi]^d$.

We note that $E(k) \geqslant \gamma |k|^2$ for $k \in [-\pi, \pi]^d$ with $\gamma > 0$ and that, for the considered values of α, $E(k)$ and its first partial derivatives are in $L^2([-\pi, \pi]^d)$.

Given $\varepsilon > 0$ and Λ finite, $\Lambda \subset \mathbb{Z}^d$, we set, as in ref. 1,

$$f_{\varepsilon, \Lambda}(x) = \frac{1}{c_\varepsilon(0)}\left[c_\varepsilon(x) + h_{\varepsilon, \Lambda}(x)\right] \tag{2.17}$$

where

$$c_\varepsilon(x) = \int_{B_d} \frac{dk}{(2\pi)^d} \frac{\cos(k \cdot x)}{E(k) + \varepsilon}, \qquad \text{with} \quad B_d = [-\pi, \pi]^d \tag{2.18}$$

and

$$c_\varepsilon(0) - c_\varepsilon(x) = \int_{B_d} \frac{dk}{(2\pi)^d} \frac{1 - \cos(k \cdot x)}{E(k) + \varepsilon} \qquad \text{for} \quad x \in \Lambda \tag{2.19}$$

$h_{\varepsilon, \Lambda}(x) = 0$ otherwise.

For the first term on the rhs of (2.14) we have, by aplying Parseval's formula,

$$\sum_{x,\, y \in C_{n_i}} [f_{\varepsilon,\Lambda}(x) - f_{\varepsilon,\Lambda}(y)]^2 \frac{1}{|x-y|^{2\alpha}} = \int_{B_d} \frac{dk}{(2\pi)^d} |\tilde{f}_{\varepsilon,\Lambda}(k)|^2 E(k) \quad (2.20)$$

where, as in the following, given a function $f: \mathbb{Z}^d \to \mathbb{C}$, $\tilde{f}$ is the Fourier transform of f

$$\tilde{f}(k) = \sum_{x \in \mathbb{Z}^d} f(x) \exp(ik \cdot x)$$

We can estimate $|\tilde{f}_{\varepsilon,\Lambda}(k)|^2$ by

$$|\tilde{f}_{\varepsilon,\Lambda}(k)|^2 \leqslant \frac{1}{c_\varepsilon(0)^2} [2|\tilde{c}_\varepsilon(k)|^2 + 2|\tilde{h}_{\varepsilon,\Lambda}(k)|^2] \quad (2.21)$$

For $\tilde{h}_{\varepsilon,\Lambda}(k)$ the following estimate holds:

$$|\tilde{h}_{\varepsilon,\Lambda}(k)| \leqslant C \operatorname{diam}(\Lambda)^{2+d} \quad (2.22)$$

with a constant C independent of Λ and ε. Indeed,

$$|h_{\varepsilon,\Lambda}(x)| = \left| \int_{B_d} \frac{dk}{(2\pi)^d} \frac{1-\cos(k \cdot x)}{E(k)+\varepsilon} \right| \leqslant \frac{|x|^2}{2} \int_{B_d} \frac{dk}{(2\pi)^d} \frac{k^2}{E(k)} \leqslant C \frac{|x|^2}{2} \quad (2.23)$$

for $x \in \Lambda$, and, consequently,

$$|\tilde{h}_{\varepsilon,\Lambda}(k)| \leqslant \sum_{x \in \Lambda} \frac{C|x|^2}{2} \leqslant C \operatorname{diam}(\Lambda)^{d+2} \quad (2.24)$$

The rhs of (2.20) can therefore be bounded by

$$\frac{1}{c_\varepsilon(0)^2} \left[\left(\int_{B_d} \frac{dk}{(2\pi)^d} \frac{1}{E(k)+\varepsilon} \right) + C \operatorname{diam}(\Lambda)^{2d+4} \right] \leqslant \frac{C}{c_\varepsilon(0)} + \frac{C \operatorname{diam}(\Lambda)^{2d+4}}{c_\varepsilon(0)^2} \quad (2.25)$$

Let us consider now the second term on the rhs of (2.14). We have

$$\sum_{x \in C_{n_i}} f^2_{\varepsilon,\Lambda}(x) \sum_{y \in C_{\bar{n}_i} \setminus C_{n_i}} \frac{C}{|x-y|^{2\alpha}} \leqslant C \sum_{x \in C_{n_i}} f^2_{\varepsilon,\Lambda}(x)(n_i - |x|)^{d-2\alpha} \quad (2.26)$$

Let now p_i be an integer with $0 < p_i < n_i$. We can bound the rhs of (2.26) by

$$C \sum_{x \in C_{p_i}} c^2_\varepsilon(x)(n_i - p_i)^{d-2\alpha} + C p_i^{-2} \sum_{x \in \mathbb{Z}^d} |x|^2 c^2_\varepsilon(x)$$
$$+ C[n_i - \operatorname{diam}(\Lambda)]^{d-2\alpha} \operatorname{diam}(\Lambda)^{2d+4}$$

where we have used the bound (2.23) on $h_{\varepsilon,\Lambda}(x)$. Since the first partial derivatives of $E(k)$ are in $L_2([-\pi,\pi]^d)$, we have that $\sum_{x\in\mathbb{Z}^d}|x|^2c_\varepsilon^2(x)$ is finite as long as $\varepsilon>0$. By putting together the estimates (2.25) and (2.27) and noticing that, for the considered values of α, $c_\varepsilon(0)$ tends to infinity as ε tends to zero,[1] we see that we can find sequences $\varepsilon_i\to 0$, $\tilde{n}_i\to\infty$, $p_i\to\infty$, $C_{n_i}\uparrow\mathbb{Z}^d$, so that (2.15) is verified. This implies that (2.16) is verified for every local observable. ▮

REFERENCES

1. C. A. Bonato, J. F. Perez, and A. Klein, *J. Stat. Phys.* **29**:159–175 (1982).
2. M. Campanino, E. Olivieri, and A. C. D. v. Enter, *Commun. Math. Phys.* **1987**:241–255 (1987).
3. A. C. D. v. Enter, *J. Stat. Phys.* **41**:315 (1985).
4. A. C. D. v. Enter and J. Fröhlich, *Commun. Math. Phys.* **98**:425–432 (1985).
5. P. Picco, *J. Stat. Phys.* **32**:627–648 (1983).